Research Article *Open Access*

Journal of Modern Classical Physics & Quantum Neuroscience

ISSN: 3068-4196

DOI: doi.org/10.63721/26JPQN0198

Prime Multiplication as a Physical Organizing Principle: Topological Coarse-Graining, Thermal Fixed Points, and Finite-Size Scaling

Phan Thành Trung

Independent Researcher, Vietnam

Citation: *Phan Thành Trung (2026) Prime Multiplication as a Physical Organizing Principle: Topological Coarse-Graining, Thermal Fixed Points, and Finite-Size Scaling. J. of Mod Phy & Quant Neuroscience 2(5), 1-7. WMJ/JPQN-198*

Abstract

Multiplicative arithmetic provides a natural composition law, but its role as an organizing principle for physical observables is largely unexplored. We construct an exactly reducible quantum-statistical ensemble in which the same two distinct primes determine a composite excitation energy and a topological Wilson observable. A two-mode state (p,q) has energy ε ln(pq) and is assigned the torus knot T(p,q) in SU (2) Chern-Simons theory at level k. At fixed level k, the Wilson value depends only on p and q modulo m_k = 2(k+2). This yields an exact coarse-graining from an unbounded prime-labelled state space to a finite congruence-sector alphabet and reduces the thermal expectation to a finite quadratic form in partial prime-zeta functions. The resulting nonlinear thermal observable has analytic high- and low-temperature endpoint values, while a finite-register energy-topology contrast obeys a provable asymptotic null limit at fixed k. Residue compression evaluates 78,498 prime modes, representing 3,080,928,753 implicit unordered pairs, without pair enumeration. The construction therefore provides a solvable arithmetic-topological complex system in which prime multiplication acts simultaneously at the levels of composition, energy, and topology. It is an engineered programmable model rather than a microscopic theory of matter, and it supplies falsifiable finite-size and thermal signatures for future tests of arithmetic organization in physical systems.

***Corresponding author**: Phan Thành Trung, Independent Researcher, Vietnam.

Submitted: 19.09.2026 **Accepted**: 25.09.2026 **Published**: 07.10.2026

Keywords: Prime Multiplication, Arithmetic-Topological Dynamics, Thermal Fixed Points, Finite-Size Scaling, Congruence Reduction, Chern-Simons Theory, Complex Quantum Systems

Introduction

Arithmetic Composition and Topological Organization

Nonlinear and complex systems often acquire tractability when a large microscopic state space admits an exact reduced description. Here the reduction is arithmetic. Chern-Simons gauge theory relates non-Abelian Wilson loops to knot invariants: for SU(2) at level k, a normalized and framing-corrected fundamental Wilson expectation gives the Jones polynomial at a root of unity [1-3]. The same observables are operationally accessible through quantum simulation, quantum approximation of Jones values, and interferometric probes of non-Abelian braiding [4-6].

Arithmetic quantum systems supply a complementary composition law. Assigning an energy proportional to ln p to a prime-labelled mode converts integer multiplication into addition of excitation energies, while arithmetic-topological quantum statistical systems have been constructed from knot semigroups [7-10]. Our question is more restrictive and more directly testable: what happens when the same multiplicative factors simultaneously determine the composite energy and the topology of a quantum observable? The resulting hierarchy links microscopic arithmetic modes, finite congruence sectors, and macroscopic thermal statistics.

Here we impose that joint rule in the square-free two-prime sector:

$$pq \longmapsto \left(E_{pq} = \varepsilon \ln(pq),\ T(p,q),\ V_{T(p,q)}(t_k)\right). \quad (1)$$

Unique factorization is essential. For distinct primes $p < q$, the product $N = pq$ uniquely recovers the unordered winding pair $\{p, q\}$; a generic coprime product does not.

Prime Hamiltonian and Wilson Observable

Let each prime label a hard-core mode, with $\hat{n}_p \in \{0,1\}$, and define

$$\hat{H}_P = \varepsilon \sum_{p\in\mathbb{P}} (\ln p)\hat{n}_p. \quad (2)$$

In the two-excitation space

$$\mathcal{H}_2 = \mathbf{span}\{|p,q\rangle : p < q\},$$

we have

$$\hat{H}_P|p,q\rangle = \varepsilon \ln(pq)|p,q\rangle. \quad (3)$$

Writing $s = \beta\varepsilon$, the restricted Gibbs state is

$$\hat{\rho}_2(s) = \frac{1}{Z_2(s)} \sum_{p<q} (pq)^{-s} |p,q\rangle\langle p,q|, \quad (4)$$

with

$$Z_2(s) = \sum_{p<q} (pq)^{-s} = \frac{P(s)^2 - P(2s)}{2}. \quad (5)$$

This state is trace class precisely for real $s > 1$ [11].

For

$$t_k = \exp\left[\frac{2\pi i}{k+2}\right],$$

define the normalized, framing-corrected Wilson value

$$w_k(p,q) = V_{T(p,q)}(t_k). \tag{6}$$

In the convention $V_\circ = 1$, the torus-knot Jones polynomial used here is

$$V_{T(p,q)}(t) = t^{\frac{(p-1)(q-1)}{2}} \frac{1 - t^{p+1} - t^{q+1} + t^{p+q}}{1 - t^2}. \tag{7}$$

The prime-conditioned normal operator and its measurable Hermitian quadratures are

$$\hat{W}_k = \sum_{p<q} w_k(p,q)|p,q\rangle\langle p,q|, \tag{8}$$

$$\hat{X}_k = \frac{\hat{W}_k + \hat{W}_k^\dagger}{2}, \qquad \hat{Y}_k = \frac{\hat{W}_k - \hat{W}_k^\dagger}{2i}. \tag{9}$$

The gauge connection remains the dynamical variable of the Chern-Simons path integral; no prime-dependent matrix is substituted for it.

Exact Congruence Coarse-Graining

Set

$$m_k = 2(k+2), \qquad P_a(s;m_k) = \sum_{\substack{p\in\mathbb{P} \\ p\equiv a \pmod{m_k}}} p^{-s}. \tag{10}$$

Equation (7) is unchanged under $p \mapsto p + m_k$ or $q \mapsto q + m_k$: the four numerator powers are periodic modulo $k+2$, while the prefactor exponent changes by an integer multiple of $k+2$. Hence there is a

symmetric finite kernel $W_{ab}^{(k)}$ such that

$$w_k(p,q) = W^{(k)}_{[p]_{m_k},[q]_{m_k}}. \tag{11}$$

Consequently, the thermal Wilson expectation has the exact finite representation

$$A_k(s) = \mathrm{Tr}\left[\hat{\rho}_2(s)\hat{W}_k\right] = \frac{1}{2Z_2(s)} \sum_{a,b=0}^{m_k-1} W_{ab}^{(k)} Q_{ab}(s), \tag{12}$$

where

$$Q_{ab}(s) = P_a(s;m_k)P_b(s;m_k) - \delta_{ab}P_a(2s;m_k). \quad (13)$$

The subtraction removes the forbidden diagonal $p = q$. At any prime cutoff X, Eq. (12) remains exact after replacing Pa by its finite partial sum. Thus, $W_{ab}^{(k)}$ has at most

$$\frac{m_k(m_k+1)}{2}$$

distinct eigenvalues, regardless of the number of prime modes. This is a bound on the spectral alphabet, not a finite-rank assertion.
The closed formula also gives

$$|w_k(p,q)| \le C_k = \frac{4}{|1-t_k^2|}.$$

Therefor $A_k(s)$ is absolutely convergent for $s > 1$. If $A_k^{(X)}$ retains $p, q \le X$, then

$$\left|A_k^{(X)}(s) - A_k(s)\right| \le \frac{2C_kP(s)}{Z_2(s)}\frac{X^{1-s}}{s-1}. \quad (14)$$

The proof and a sharper exact tail expression are given in the Supplemental Material.

Nonlinear Thermal Flow and Endpoint Fixed Points

Let $U(m_k)$ be the unit residues modulo m_k. The prime number theorem in arithmetic progressions implies, for fixed m_k,

$$P_a(s;m_k) = \frac{1}{\varphi(m_k)}\ln\frac{1}{s-1} + O(1), \qquad a \in U(m_k), \quad (15)$$

as $s \to 1^+$. Nonunit classes contain only the finitely many primes dividing mk. Inserting Eq. (15) into Eq. (12) yields the exact high-temperature boundary value

$$\lim_{s\to 1^+} A_k(s) = \frac{1}{\varphi(m_k)^2}\sum_{a,b\in U(m_k)} W_{ab}^{(k)}. \quad (16)$$

At the opposite endpoint, the smallest pair product is $2 \cdot 3$, followed by $2 \cdot 5$, giving

$$A_k(s) = w_k(2,3) + O\left[\left(\frac{3}{5}\right)^s\right], \qquad s \to \infty. \quad (17)$$

The one-parameter thermal observable therefore flows between a unit-residue Wilson average and the trefoil Wilson value, rather than between fitted numerical parameters. We use the term thermal fixed point for these exact endpoint values of the s-flow; no renormalization-group fixed point is assumed. The nonlinearity arises

from the normalized quadratic dependence on partial prime-zeta functions in Eq. (12).

Finite-Size Scaling and the Asymptotic Null Limit

For a finite register, let $epq = \ln(pq)$ and define

$$\chi_k^{(X)} = \left[\mathrm{Corr}(e, \mathrm{Re}\, w_k)^2 + \mathrm{Corr}(e, \mathrm{Im}\, w_k)^2\right]^{1/2}. \quad (18)$$

Because W_k is constant on residue-pair sectors, every covariance in Eq. (18) depends only on the mode counts and the sums of $\ln p$ inside those sectors. Permuting energies within each residue class therefore leaves $\chi_k^{(x)}$ exactly unchanged. This proves that the contrast is congruence mediated; it cannot be interpreted as prime-specific information beyond the residue structure at fixed.

The same reduction supplies its asymptotic behavior. For fixed m_k, the prime number theorem in arithmetic progressions makes the counts and logarithmic-energy moments equidistributed among the unit classes. Exceptional primes have vanishing weight in the uniform finite-register pair ensemble. It follows that

$$\lim_{X\to\infty} \chi_k^{(X)} = 0 \qquad \text{for every fixed } k. \quad (19)$$

Finite-size peaks therefore must not be extrapolated into a physical phase. Instead they form a controlled finite-size scaling diagnostic whose exact null limit separates residue-mediated organization from genuinely prime-specific information. Between X=100 and X=1,000,000, the reported contrasts decrease systematically at k=2,4,8,10; the complete cutoff series and seeded controls are supplied with the reproducibility package.

Reduced-State Measurement Protocol

A finite simulator prepares two distinct hard-core modes with Gibb's weight

$$\frac{(pq)^{-s}}{Z_2(s)}.$$

It reversibly computes p,q modulo mk and controls an ancilla readout using the finite table $W_{ab}^{(k)}$. Separate Hadamard-type measurements give $\langle \hat{X}_k \rangle_s$ and $\langle \hat{Y}_k \rangle_s$.The topological lookup contains at most $m_k(m_k+1)/2$ complex values and is independent of X; only state preparation and modular label processing grow with the register.

If each shot is normalized to a range bounded by C_k, Hoeffding's inequality gives

$$N_{\mathrm{shot}} \geq \frac{2C_k^2}{\delta^2} \ln\frac{2}{\alpha}, \quad (20)$$

for additive error δ and failure probability at most α. A full braid implementation may instead condition

$$(\sigma_1 \cdots \sigma_{p-1})^q$$

on $|p, q\rangle$; the compressed protocol measures the same root-of-unity observable without claiming to reproduce the complete braid dynamics.

The numerical implementation validates direct pair enumeration against Eq. (12) at $X = 500$ to a maximum absolute error below 8.5×10^{-16}. It then evaluates 78,498 prime modes - representing 3,080,928,753 unordered pairs - without constructing a pair array. Periodicity and residue-permutation invariance is verified to floating-point precision. Source code, complete kernels, endpoint values, cutoff series, seeded global-permutation controls, and validation logs accompany this study.

Discussion and Outlook

The physical claim is deliberately bounded but nontrivial. The Hamiltonian in Eq. (2) is an engineered programmable spectrum, not a microscopic model of matter, and the construction does not identify prime numbers with particles or fields. Its physical content is operational: prime multiplication simultaneously sets a composite energy, a torus-knot label, a finite congruence sector of a Wilson observable, and a measurable thermal response. Exact coarse-graining then shows how an unbounded arithmetic microstate space collapses to a finite topological spectral alphabet at fixed gauge level.

This hierarchy is the central complex-systems result: arithmetic composition → logarithmic energy → congruence sector → topological observable → thermal statistics. Its analytically known endpoints and asymptotic null theorem make it falsifiable at finite register size rather than merely metaphorical. The model therefore provides a controlled setting in which the physical consequences of multiplicative arithmetic can be isolated from arbitrary numerical labelling.

A broader empirical question is whether multiplicative descriptors of nuclear or elemental identity exhibit analogous reduced-sector organization. The present work does not establish that prime factors constitute nuclei or generate chemical elements. Testing such a connection requires independent nuclear or atomic data, leakage-safe validation, and matched non-prime controls. The exact ensemble developed here supplies a mathematical and computational template for that next experiment.

Conclusions

We have shown that prime multiplication can serve as a physical organizing rule in an exactly solvable programmable quantum ensemble: the product pq determines additive energy while the same factors determine a torus-knot Wilson observable. At fixed Chern-Simons level, exact congruence coarse-graining compresses infinitely many prime-labelled states to a finite spectral alphabet, the thermal flow has analytic endpoint values, and the finite-register contrast has a provable asymptotic null limit. These results separate robust arithmetic-topological structure from finite-size coincidence and provide direct observables for simulation. The broader role of multiplicative arithmetic in physical composition remains an empirical question, now posed in a form that can be quantitatively tested.

Data and Code

No external data were used. The source code, deterministic numerical tables, validation report, and reproducibility instructions are available at the repository below. Generative-AI assistance was used during code development and debugging; all reported numerical outputs were regenerated with deterministic scripts and checked against the exact congruence formulas and direct pair enumeration at small cutoffs.
https://doi.org/10.5281/zenodo.21876575

Credit Authorship Contribution Statement

Phan Thành Trung: Conceptualization, Methodology, Formal analysis, Investigation, Software, Validation, Visualization, Writing - original draft, Writing - review & editing.

Funding

This research received no external funding.

Declaration of Competing Interest

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments
The author acknowledges the foundational work on prime spectra, arithmetic progressions, knot invariants, quantum algorithms, and Chern-Simons theory on which this construction is based.

Declaration of Generative AI and AI-assisted Technologies in the Manuscript Preparation Process
During the preparation of this work the author used OpenAI ChatGPT and Codex to assist with manuscript drafting and language refinement, computational-code development and debugging, and the presentation of mathematical content. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

References
1. Witten E (1989) Quantum field theory and the Jones polynomial. Communications in Mathematical Physics 121: 351-399.
2. Jones VFR (1985) A polynomial invariant for knots via von Neumann algebras. Bulletin of the American Mathematical Society 12: 103-111.
3. Rosso M, Jones VFR (1993) On the invariants of torus knots derived from quantum groups. Journal of Knot Theory and Its Ramifications 2: 97-112.
4. Freedman MH, Kitaev A, Wang Z (2002) Simulation of topological field theories by quantum computers. Communications in Mathematical Physics 227: 587-603.
5. Aharonov D, Jones V, Landau Z (2009) A polynomial quantum algorithm for approximating the Jones polynomial. Algorithmica 55: 395-421.
6. Bonderson P, Shtengel K, Slingerland JK (2006) Probing non-abelian statistics with quasiparticle interferometry. Physical Review Letters 97: 016401.
7. Spector D (1989) Multiplicative functions, Dirichlet convolution, and quantum systems. Physics Letters A 140: 311-314.
8. Spector D (1990) Supersymmetry and the Möbius inversion function. Communications in Mathematical Physics 127: 239-252.
9. Schumayer D, Hutchinson DAW (2011) Colloquium: Physics of the Riemann hypothesis. Reviews of Modern Physics 83: 307-330.
10. Marcolli M, Xu Y (2017) Quantum statistical mechanics in arithmetic topology. Journal of Geometry and Physics 114: 153-183.
11. Fröberg CE (1968) On the prime zeta function. BIT Numerical Mathematics 8: 187-202.
12. Davenport H (2000) Multiplicative Number Theory, 3rd ed. Springer, New York. (Revised by Montgomery HL).