



Refined Uncertainty Relation For Q-Commutator

Kenjiro Yanagi

Emeritus Professor of Yamaguchi University, Japan

*Citation: Kenjiro Yanagi (2025) Refined Uncertainty Relation For Q-Commutator.
J.of Mod Phy & Quant Neuroscience 1(4), 1-07. WMJ/JPQN-136*

Abstract

We show that Robertson uncertainty relation can be refined for q -commutator which is defined by $[A, B]_q = AB - qBA$, where A, B are self-adjoint operators and $q \in \mathbb{R}$. The coefficient is represented by the eigenvalues of state ρ .

***Corresponding author:** Kenjiro Yanagi, Emeritus Professor of Yamaguchi University, Japan.

Submitted: 16.10.2025

Accepted: 22.10.2025

Published: 02.12.2025

Introduction- Historical background

In quantum mechanical system, the expectation value of an observable A in a quantum state ρ is expressed by $Tr[\rho A]$. It is natural that the variance for a quantum state ρ and an observable A is defined by $V_\rho(A) = Tr[\rho A^2] - (Tr[\rho A])^2 = Tr[\rho A_0^2]$, where $A_0 = A - (Tr[\rho A])I$. It is well known that Robertson uncertainty relation is represented by the commutator $[A, B] = AB - BA$ in the following.

Theorem 1.1. Let ρ be a density operator and A, B be self-adjoint operators. Then the following uncertainty relation holds.

$$\frac{1}{4}|Tr[\rho[A, B]]|^2 \leq V_\rho(A)V_\rho(B).$$

It is easy to show the following uncertainty relation for q -commutator $[A, B]_q = AB - qBA$.

Theorem 1.2. Let ρ be a density operator and A, B be self-adjoint operators. Then the following uncertainty relation holds for $q \in \mathbb{R}$.

$$\frac{1}{(1 + |q|)^2} |Tr[\rho[A_0, B_0]_{|q|}]|^2 \leq V_\rho(A)V_\rho(B).$$

In this paper we have an extension of Theorem 1.2 by using the maximal eigenvalue and minimal eigenvalue of ρ under finite dimensional case.

2. Q-commutator

Let $M_n(\mathbb{C}), M_{n,sa}(\mathbb{C})$ and $M_{n,l}(\mathbb{C})$ be the set of all n dimensional complex matrices, the set of all n dimensional self-adjoint matrices and the set of all n dimensional density matrices. For $A, B \in M_{n,sa}(\mathbb{C})$, we define commutator and anti-commutator by $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$, respectively. Q-commutator and q-anti-commutator are defined by $[A, B]_q = AB - qBA$ and $\{A, B\}_q = AB + qBA$, respectively, where $q \in \mathbb{R}$ and $A, B \in M_{n,sa}(\mathbb{C})$. Q-commutator is a generalization of commutator $[A, B]$. Now we state the main theorem.

Theorem 2.1. Let $\rho = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i|$ be the spectral decomposition satisfying $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then we have the following refined uncertainty relations.

(1) If $|q| \leq 1$, then

$$\frac{(\lambda_n + |q|\lambda_1)^2}{(1 + |q|)^2(\lambda_n - |q|\lambda_1)^2} |Tr[\rho[A_0, B_0]_{|q|}]|^2 \leq V_\rho(A)V_\rho(B).$$

(2) If $|q| > 1$, then

$$\begin{aligned} & \frac{q^2(|q|\lambda_n + \lambda_1)^2}{(1 + |q|)^2(|q|\lambda_n - \lambda_1)^2} |Tr[\rho[A_0, B_0]_{1/|q|}]|^2 \\ &= \frac{(|q|\lambda_n + \lambda_1)^2}{(1 + |q|)^2(|q|\lambda_n - \lambda_1)^2} |Tr[\rho[B_0, A_0]_{|q|}]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned}$$

We need some lemmas in order to prove the main result.

Lemma 2.1. If $|q| \leq 1$, then $G(t) = \left(\frac{t-|q|}{t+|q|}\right)^2$ is monotone increasing for $t \geq 1$.

Proof. $G'(t) = \frac{4|q|(t-|q|)}{(t+|q|)^3} \geq 0$. □

Lemma 2.2. If $|q| \leq 1$, then $F(t) = (1 + |q|t)^2(t - |q|)^2 - (1 - |q|t)^2(t + |q|)^2 \geq 0$ for $t \geq 1$.

That is

$$\left(\frac{1 - |q|t}{1 + |q|t}\right)^2 \leq \left(\frac{t - |q|}{t + |q|}\right)^2.$$

Proof. $F'(t) = 4|q|(1 - |q|^2)(3t^2 - 1) \geq 0$. Since $F(1) = 0$, we have $F(t) \geq 0$ for $t \geq 1$ □

Proof of Theorem 2.1. Let $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle\phi_j|B_0|\phi_i\rangle$.

(1) Since $Tr[\rho[A_0, B_0]_{|q|}] = \sum_{i,j} (\lambda_i - |q|\lambda_j)a_{ij}b_{ji}$,

$$|Tr[\rho[A_0, B_0]_{|q|}]| \leq \sum_{i,j} |\lambda_i - |q|\lambda_j| |a_{ij}| |b_{ji}|.$$

By Schwarz inequality

$$\begin{aligned} |Tr[\rho[A_0, B_0]_{|q|}]|^2 &\leq \left(\sum_{i,j} |\lambda_i - |q|\lambda_j| |a_{ij}| |b_{ji}| \right)^2 \\ &\leq \sum_{i,j} |\lambda_i - |q|\lambda_j| |a_{ij}|^2 \sum_{i,j} |\lambda_i - |q|\lambda_j| |b_{ji}|^2. \end{aligned}$$

Then

$$(2.1) \quad \frac{1}{(\lambda_n - |q|\lambda_1)^2} |Tr[\rho[A_0, B_0]_{|q|}]|^2 \leq \sum_{i,j} \left| \frac{\lambda_i - |q|\lambda_j}{\lambda_n - |q|\lambda_1} \right| |a_{ij}|^2 \cdot \sum_{i,j} \left| \frac{\lambda_i - |q|\lambda_j}{\lambda_n - |q|\lambda_1} \right| |b_{ji}|^2.$$

In the case of $\lambda_i \geq \lambda_j$, since $1 \leq \lambda_i/\lambda_j \leq \lambda_n/\lambda_1$, it follows the following from Lemma 2.1

$$\left(\frac{\lambda_i/\lambda_j - |q|}{\lambda_i/\lambda_j + |q|} \right)^2 \leq \left(\frac{\lambda_n/\lambda_1 - |q|}{\lambda_n/\lambda_1 + |q|} \right)^2.$$

In the case of $\lambda_i < \lambda_j$, since $1 < \lambda_j/\lambda_i \leq \lambda_n/\lambda_1$, it follows the following from Lemma 2.2 and Lemma 2.1

$$\left(\frac{1 - |q|(\lambda_j/\lambda_i)}{1 + |q|(\lambda_j/\lambda_i)} \right)^2 \leq \left(\frac{\lambda_j/\lambda_i - |q|}{\lambda_j/\lambda_i + |q|} \right)^2 \leq \left(\frac{\lambda_n/\lambda_1 - |q|}{\lambda_n/\lambda_1 + |q|} \right)^2.$$

In both cases

$$\left(\frac{\lambda_i - |q|\lambda_j}{\lambda_i + |q|\lambda_j} \right)^2 \leq \left(\frac{\lambda_n - |q|\lambda_1}{\lambda_n + |q|\lambda_1} \right)^2.$$

That is

$$\left| \frac{\lambda_i - |q|\lambda_j}{\lambda_n - |q|\lambda_1} \right| \leq \left| \frac{\lambda_i + |q|\lambda_j}{\lambda_n + |q|\lambda_1} \right|.$$

By (2.1) we have

$$\frac{1}{(\lambda_n - |q|\lambda_1)^2} |Tr[\rho[A_0, B_0]_{|q|}]|^2 \leq \sum_{i,j} \left| \frac{\lambda_i + |q|\lambda_j}{\lambda_n + |q|\lambda_1} \right| |a_{ij}|^2 \cdot \sum_{i,j} \left| \frac{\lambda_i + |q|\lambda_j}{\lambda_n + |q|\lambda_1} \right| |b_{ji}|^2.$$

Since $|a_{ij}| = |a_{ji}|$, we have

$$\begin{aligned} &\sum_{i,j} |\lambda_i + |q|\lambda_j| |a_{ij}|^2 = \sum_{i,j} \lambda_i |a_{ij}|^2 + |q| \sum_{i,j} \lambda_j |a_{ij}|^2 \\ &= \sum_{i,j} \lambda_i |a_{ij}|^2 + |q| \sum_{i,j} \lambda_i |a_{ji}|^2 = \sum_{i,j} \lambda_i |a_{ij}|^2 + |q| \sum_{i,j} \lambda_i |a_{ij}|^2 \\ &= (1 + |q|) \sum_{i,j} \lambda_i |a_{ij}|^2 = (1 + |q|) V_\rho(A). \end{aligned}$$

Then

$$\frac{(\lambda_n + |q|\lambda_1)^2}{(1 + |q|)^2 (\lambda_n - |q|\lambda_1)^2} |Tr[\rho[A_0, B_0]_{|q|}]|^2 \leq V_\rho(A) V_\rho(B).$$

(2) Since $1/|q| < 1$, we have the following by (1).

$$\begin{aligned} & \frac{(\lambda_n + (1/|q|)\lambda_1)^2}{(1 + 1/|q|)^2(\lambda_n - (1/|q|)\lambda_1)^2} |Tr[\rho[A_0, B_0]_{1/|q|}]|^2 \\ &= \frac{q^2(|q|\lambda_n + \lambda_1)^2}{(1 + |q|)^2(|q|\lambda_n - \lambda_1)^2} |Tr[\rho[A_0, B_0]_{1/|q|}]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned}$$

Since

$$Tr[\rho[A_0, B_0]_{1/|q|}] = -\frac{1}{|q|} Tr[\rho[B_0, A_0]_{|q|}],$$

we have

$$\frac{(|q|\lambda_n + \lambda_1)^2}{(1 + |q|)^2(|q|\lambda_n - \lambda_1)^2} |Tr[\rho[B_0, A_0]_{|q|}]|^2 \leq V_\rho(A)V_\rho(B).$$

□

We obtain the following corollary.

Corollary 2.1. *We have the following refined uncertainty relations under five conditions.*

(1) If $0 < q \leq 1$, then

$$\frac{(\lambda_n + q\lambda_1)^2}{(1 + q)^2(\lambda_n - q\lambda_1)^2} |Tr[\rho[A_0, B_0]_q]|^2 \leq V_\rho(A)V_\rho(B).$$

(2) If $q > 1$ then

$$\begin{aligned} & \frac{q^2(q\lambda_n + \lambda_1)^2}{(1 + q)^2(q\lambda_n - \lambda_1)^2} |Tr[\rho[A_0, B_0]_{1/q}]|^2 \\ &= \frac{(q\lambda_n + \lambda_1)^2}{(1 + q)^2(q\lambda_n + \lambda_1)^2} |Tr[\rho[B_0, A_0]_q]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned}$$

(3) If $q = 0$, then $|Tr[\rho A_0 B_0]|^2 \leq V_\rho(A)V_\rho(B)$.

(4) If $-1 \leq q < 0$, then

$$\begin{aligned} & \frac{(\lambda_n - q\lambda_1)^2}{(1 - q)^2(\lambda_n + q\lambda_1)^2} |Tr[\rho[A_0, B_0]_{-q}]|^2 \\ &= \frac{(\lambda_n - q\lambda_1)^2}{(1 - q)^2(\lambda_n + q\lambda_1)^2} |Tr[\rho\{A_0, B_0\}_q]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned}$$

(5) If $q < -1$, then

$$\begin{aligned} & \frac{q^2(\lambda_1 - q\lambda_n)^2}{(1 - q)^2(\lambda_1 + q\lambda_n)^2} |Tr[\rho[A_0, B_0]_{-1/q}]|^2 \\ &= \frac{q^2(\lambda_1 - q\lambda_n)^2}{(1 - q)^2(\lambda_1 + q\lambda_n)^2} |Tr[\rho\{A_0, B_0\}_{1/q}]|^2 \\ &= \frac{(\lambda_1 - q\lambda_n)^2}{(1 - q)^2(\lambda_1 + q\lambda_n)^2} |Tr[\rho\{B_0, A_0\}_q]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned}$$

We obtain the result by [3]. This is the case of $q = 1$ in Theorem 2.1 and Corollary 2.1.

Corollary 2.2.

$$\frac{(\lambda_n + \lambda_1)^2}{4(\lambda_n - \lambda_1)^2} |Tr[\rho[A_0, B_0]]|^2 = \frac{(\lambda_n + \lambda_1)^2}{4(\lambda_n - \lambda_1)^2} |Tr[\rho[A, B]]|^2 \leq V_\rho(A)V_\rho(B).$$

Remark 2.1. Let $\rho = \frac{1}{n}I$ be the maximally mixed state. If $|q| < 1$, then

$$\frac{1}{(1 - |q|)^2} |Tr[\rho[A_0, B_0]_{|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

If $|q| > 1$, then

$$\frac{1}{(1 + |q|)^2} |Tr[\rho[B_0, A_0]_{|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

These are the same as $|Tr[A_0B_0]|^2 \leq Tr[A_0^2]Tr[B_0^2]$.

Remark 2.2. Let ρ be non-faithfull state. Since $\lambda_1 = 0$, we have the same result as Theorem 1.2. Because in the case of $|q| \leq 1$,

$$\frac{1}{(1 + |q|)^2} |Tr[\rho[A_0, B_0]_{|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

Then we can rewrite the following

$$\frac{1}{(1 + |q|)^2} |Tr[\rho[B_0, A_0]_{|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

Then in the case of $|q| > 1$, we have

$$\frac{|q|^2}{(1 + |q|)^2} |Tr[\rho[B_0, A_0]_{1/|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

Therefore

$$\frac{1}{(1 + |q|)^2} |Tr[\rho[A_0, B_0]_{|q}|]|^2 \leq V_\rho(A)V_\rho(B).$$

Acknowledgement

The author would like to thank Prof. G. Kimura for giving me interesting results.

References

1. R J Finkerstein (1998) q-uncertainty relations, J. Modern Physics, A 13: 1795-1803.
2. W Heisenberg (1927) über den anschaulichen Inhalt der quantummechanischen Kinematik und Mechanik Zeitschrift für Physik 43: 172-198.
3. G Kimura, A Mayumi, H Yamashita (2025), Tight generalization of Robertson-type uncertainty relations, arXiv:2505.19861v1 [quant-ph].
4. K Yanagi (2024), Uncertainty relation for q-commutator, Linear and Nonlinear Analysis 10: 124126.

Copyright: ©2025 Kenjiro Yanagi. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.